
GROUP THEORETIC FORMULATION OF COMPLEMENTARITY

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We generalize Bohr's complementarity principle for wave and particle properties to arbitrary quantum systems. We begin by noting that a particle-like state is represented by a spatially-localized wave function and its narrow probability density is displaced by spatial translations. In contrast a wave-like state is represented by a spatially-delocalized wave function and the corresponding broad position probability density is invariant to spatial translations. The wave-particle dichotomy can therefore be seen as a competition between displacement and invariance of the state with respect to spatial translations. We generalize this dichotomy to arbitrary quantum systems with finite dimensional Hilbert spaces as follows. We use arbitrary finite symmetry groups to represent transformations of the quantum system. The symmetry (i.e. invariance) or asymmetry (i.e. displacement) of a given state with respect to transformations of the group are identified with the generalized wave and particle nature, respectively. We adopt a measure of wave and particle properties based on the amount of information that can be encoded in the symmetric and asymmetric parts of the state.

1. INTRODUCTION

Bohr's complementarity principle is a defining feature of quantum physics [1]. In essence it represents the dichotomy between the particle and wave nature of mechanical objects; the particle properties are typically symbolized by well-defined position and the wave properties by well-defined momentum. More recent work has attempted to quantify the wave-like and particle-like properties and study the range of properties in between the extremes of pure wave- and pure particle-like states. For example Wootters and Zurek [2] formulated an inequality for a double slit experiment that expresses a lower bound on the loss of path information (i.e. information about which slit a photon passes through) for a given sharpness of the interference pattern. Scully *et al.* [3] explored the erasure of path information and the recovery of an interference pattern using sub-ensembles conditioned on ancillary measurements. A debate regarding the application of an uncertainty principle ensued ([4] and references therein). Also Englert [5] derived an inequality for a two-way interferometer that limits the distinguishability of the outcomes of a path measurement and the visibility of the interference pattern. The related study of the simultaneous measurement of non-commuting observables also has a similarly long history [6].

Here we generalize the complementarity principle as follows. We first note that a state with particle-like properties is *displaced* (i.e. mapped) under a spatial translation to an orthogonal state, owing to its narrowness in the position representation. For example, the wave function $\psi(x) = \langle x | \psi \rangle$, where $|x\rangle$ is an eigenket of position and $|\psi\rangle$ is a general state, is mapped to $\langle x | e^{i\hat{p}\delta x/\hbar} | \psi \rangle = \langle x + \delta x | \psi \rangle = \psi(x + \delta x)$ under the spatial translation given

by $e^{i\hat{p}\delta x/\hbar}$ where \hat{p} is the position operator. The overlap $\int \psi^*(x)\psi(x+\delta x)dx$ is negligible for finite translations $\delta x > 0$ and particle-like states of the kind $|\psi\rangle \propto \int e^{-(x-x')^2/4\sigma^2}|x\rangle dx$ with a sufficiently small value of σ . In contrast, a wave-like state is essentially delocalized in the position representation so that the position probability density $P_{\text{pos}}(x) = |\langle x|\psi\rangle|^2$ is essentially “flat” and *invariant* to spatial translations. For example, $P_{\text{pos}}(x+\delta x) = |\langle x|e^{i\hat{p}\delta x/\hbar}|\psi\rangle|^2$ is approximately equal to $P_{\text{pos}}(x)$ for arbitrary translations δx and wave-like states of the kind $|\psi\rangle \propto \int e^{-(x-x')^2/4\sigma^2}|x\rangle dx$ with a sufficiently large value of σ . In other words, waves are *symmetric* (i.e. invariant) and particles are *asymmetric* (i.e. displaced) with respect to spatial translations.

Next we generalize this notion by associating “generalized” wave and particle nature with symmetry and asymmetry with respect to an arbitrary finite symmetry group, which we represent as $G = \{g_1, g_2, \dots, g_{|G|}\}$ of order $|G|$. We consider the generalized particle and wave nature of a system with state density operator $\hat{\rho}$. For convenience we call the generalized particles simply “particles”, and similarly generalized waves “waves”, and we refer to transformations by the group as “translations”. Let G have the unitary representation \hat{T}_g for $g \in G$ on the system’s Hilbert space.

2. INFORMATION THEORETIC COMPLEMENTARITY

We first consider particle nature of the state $\hat{\rho}$. Particle-like states are translated by the actions of the group $G = \{g\}$ and hence their translation

$$\hat{\rho} \mapsto \hat{\rho}_g = \hat{T}_g \hat{\rho} \hat{T}_g^\dagger \quad (1)$$

for $g \in G$ can carry information. We imagine an information theoretic scenario between two separated parties A and B as follows. Party A prepares the system in the translated state $\hat{\rho}_g$ for $g \in G$ with uniform probability $p(g) = \frac{1}{|G|}$ and sends it to B. Party B then makes a measurement on the system to estimate the value of the parameter g . We define an *information-theoretic measure of particle nature*, $N_{\text{Part}}(\hat{\rho})$, of $\hat{\rho}$ as the maximum of the mutual information between A and B over all possible measurements at B[§]. Let B make the measurement described by the Kraus operators \hat{M}_k with POM elements $\hat{\pi}_k = \hat{M}_k^\dagger \hat{M}_k$ satisfying $\sum_k \hat{\pi}_k = \hat{\mathbb{I}}$. The probability that B obtains outcome k given that the system is initially in state $\hat{\rho}_g$ is $P(k|g) = \text{Tr}(\hat{M}_k \hat{\rho}_g \hat{M}_k^\dagger)$. Note that $p(g)P(k|g) = q(k)Q(g|k)$, where $q(k) = \sum_g p(g)P(k|g)$ is the average probability of B obtaining result k , and $Q(g|k)$ is the probability the system was prepared in state $\hat{\rho}_g$ given that B obtains the outcome k . Thus we find mutual information shared by both A and B for this measurement is given by

$$I_{\text{Part}} = H(\{p_g\}) - \sum_k q(k)H(\{Q(g|k)\}) \quad (2)$$

where the $H(\{r(j)\})$ is the Shannon entropy associated with the set of probabilities $\{r(j) : j = 1, 2, \dots\}$, i.e.

$$H(\{r(j)\}) = - \sum_j r(j) \log r(j) . \quad (3)$$

The particle nature $N_{\text{Part}}(\hat{\rho})$ of $\hat{\rho}$ is the maximum of I_{Part} over all possible measurements at B. It is bounded above by Holevo’s theorem [7]:

[§]In other words, $N_{\text{Part}}(\hat{\rho})$ is the *accessible* information B has about the parameter g prepared by A.

$$N_{\text{Part}}(\hat{\rho}) \leq S(\mathcal{G}[\hat{\rho}]) - \frac{1}{|G|} \sum_{g \in G} S(\hat{\rho}_g) \quad (4)$$

where $S(\hat{\rho}) = -\text{Tr}(\hat{\rho} \ln \hat{\rho})$ is the von Neumann entropy of $\hat{\rho}$, and [10]

$$\mathcal{G}[\hat{\rho}] \equiv \frac{1}{|G|} \sum_{g \in G} \hat{T}_g \hat{\rho} \hat{T}_g^\dagger = \frac{1}{|G|} \sum_{g \in G} \hat{\rho}_g \quad (5)$$

is the average state received by B. Noting that as \hat{T}_g is unitary, $S(\hat{\rho}_g) = S(\hat{T}_g \hat{\rho} \hat{T}_g^\dagger) = S(\hat{\rho})$ for all $g \in G$ we find Eq. (4) becomes

$$N_{\text{Part}}(\hat{\rho}) \leq S(\mathcal{G}[\hat{\rho}]) - S(\hat{\rho}) . \quad (6)$$

We have previously defined the quantity $A_G(\hat{\rho}) = S(\mathcal{G}[\hat{\rho}]) - S(\hat{\rho})$ as the asymmetry of $\hat{\rho}$ with respect to the group G [8]; hence we find here that *the information-theoretic measure of particle nature is bounded by the asymmetry of $\hat{\rho}$* , which is consistent with our identification of particles with asymmetry.

We now consider the analogous information theoretic scenario for wave nature. We imagine that party A encodes information in the wave properties of the state $\hat{\rho}$, using a suitably restricted class of operations that leave the particle properties unchanged, and then sends the system to B who decodes the information using measurements. Rather than specify the kinds of operators that A can use we note that the wave properties of the state are invariant to translations \hat{T}_g for $g \in G$, and so in terms of the wave nature the state $\hat{\rho}$ is equivalent to $\hat{T}_g \hat{\rho} \hat{T}_g^\dagger$ and also to $\mathcal{G}[\hat{\rho}]$. Moreover, the latter state $\mathcal{G}[\hat{\rho}]$ is symmetric in the sense that it is invariant to translations of the group [10], i.e.

$$\hat{T}_g(\mathcal{G}[\hat{\rho}])\hat{T}_g^\dagger = \mathcal{G}[\hat{\rho}] \quad \text{for } g \in G , \quad (7)$$

and so $\mathcal{G}[\hat{\rho}]$ is devoid of any particle nature that $\hat{\rho}$ might have. Thus A can encode information using *arbitrary* operations on $\mathcal{G}[\hat{\rho}]$ and be sure that the encoding uses only wave properties of $\hat{\rho}$. This leads to an equivalent measure for wave nature as follows. Party A encodes information in the system by preparing the state

$$\hat{\rho}'_j = \hat{U}_j(\mathcal{G}[\hat{\rho}])\hat{U}_j^\dagger \quad (8)$$

with probability $p'(j) = \frac{1}{N}$ for an arbitrary set of N unitary operators $\{\hat{U}_1, \hat{U}_2, \dots, \hat{U}_N\}$. The system is then sent to B who makes a measurement to estimate the value of the parameter j . Let B's measurement be described by the Kraus operators \hat{M}'_k with POM elements $\hat{\pi}'_k = \hat{M}'_k \hat{M}'_k^\dagger$ where $\sum_k \hat{\pi}'_k = \hat{\mathbb{1}}$. The probability that B obtains outcome k for the system in state $\hat{\rho}'_j$ is given by $P'(k|j) = \text{Tr}(\hat{M}'_k \hat{\rho}'_j \hat{M}'_k^\dagger)$. Again note that $p'(j)P'(k|j) = q'(k)Q'(j|k)$ where $q'(k) = \sum_j p'(j)P'(k|j)$ is the average probability of B obtaining result k and $Q'(j|k)$ is the probability the system was prepared in state $\hat{\rho}'_j$ given that B obtains the outcome k . Thus the mutual information shared by both A and B for this measurement is given by

$$I_{\text{Wave}} = H(\{p'(j)\}) - \sum_k q'(k)H(\{Q'(j|k)\}) . \quad (9)$$

The maximum of I_{Wave} over all possible measurements at B is bounded by Holevo's theorem [7]:

$$I_{\text{Wave}}^{(\max)} \leq S\left(\sum_j p'(j)\hat{\rho}'_j\right) - \sum_j p'(j)S(\hat{\rho}'_j) . \quad (10)$$

As the operators \hat{U}_j are unitary we find $S(\hat{\rho}'_j) = S(\mathcal{G}[\hat{\rho}])$. We define an *information-theoretic measure of wave nature*, $N_{\text{Wave}}(\hat{\rho})$, of $\hat{\rho}$ as the maximum of $I_{\text{Wave}}^{(\max)}$ over all possible preparations at A. This maximum is given by the largest value of the bound on the right-hand side of Eq. (10), that is, for $\sum_j \hat{\rho}'_j = \hat{\mathbb{1}}$. Hence we have

$$N_{\text{Wave}}(\hat{\rho}) \leq \log(D) - \sum_j S(\mathcal{G}[\hat{\rho}]) \quad (11)$$

where D is the dimension of the Hilbert space. We have previously defined the quantity $W_G(\hat{\rho}) = \log(D) - S(\mathcal{G}[\hat{\rho}])$ as the symmetry of $\hat{\rho}$ with respect to G [8]. Thus *the information-theoretic measure of wave nature is bounded by the symmetry of $\hat{\rho}$* , which is consistent with our association of waves with symmetry.

Combining the two expressions (6) and (11) yields the complementarity relation:

$$N_{\text{Part}}(\hat{\rho}) + N_{\text{Wave}}(\hat{\rho}) \leq \ln(D) - S(\hat{\rho}) . \quad (12)$$

That is, *the sum of the information-theoretic measures of particle and wave natures is bounded by the maximum information that can be carried by the system*. A more extensive analysis will be reported elsewhere [11].

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